

# Stochastic perturbations of iterations of a simple, non-expanding, nonperiodic, piecewise linear, interval-map

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## Abstract

Let  $g(x) = x/2 + 17/30 \pmod{1}$ , let  $\xi_i, i = 1, 2, \dots$  be a sequence of independent, identically distributed random variables with uniform distribution on the interval  $[0, 1/15]$ , define  $g_i(x) = g(x) + \xi_i \pmod{1}$  and, for  $n = 1, 2, \dots$ , define  $g^n(x) = g_n(g_{n-1}(\dots(g_1(x))\dots))$ . For  $x \in [0, 1)$  let  $\mu_{n,x}$  denote the distribution of  $g^n(x)$ . The purpose of this note is to show that there exists a unique probability measure  $\mu$ , such that, for all  $x \in [0, 1)$ ,  $\mu_{n,x}$  tends to  $\mu$  as  $n \rightarrow \infty$ . This contradicts a claim by Lasota and Mackey from 1987 stating that the process has an asymptotic three-periodicity.

**Keywords:** convergence of distributions, random dynamical systems, stochastic perturbations of iterations, non-expanding interval maps

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## 1 Introduction

Let  $S = [0, 1)$ , let  $g : S \rightarrow S$  be defined by

$$g(x) = ax + b \pmod{1} \quad (1)$$

where

$$a = 1/2 \text{ and } b = 17/30. \quad (2)$$

Let  $\xi_n, n = 1, 2, \dots$  be a sequence of independent, identically distributed, random variables, define  $g_n : S \rightarrow S$  by

$$g_n(x) = g(x) + \xi_n \pmod{1}$$

and define  $g^{(n)} : S \rightarrow S, n = 1, 2, \dots$  recursively by

$$g^{(1)}(x) = g_1(x)$$

$$g^{(n+1)}(x) = g_{n+1}(g^{(n)}(x)), n = 1, 2, \dots$$

We write  $\xi^{(n)} = (\xi_1, \xi_2, \dots, \xi_n)$  and, if we want to emphasize  $g^{(n)}(x)$ 's dependence of  $\xi_1, \xi_2, \dots, \xi_n$ , we write

$$g^{(n)}(x) = g^{(n)}(x; \xi^{(n)}).$$

In the paper [2] from 1987, A. Lasota and M. C. Mackey considered the process  $\{g^{(n)}(x), n = 1, 2, \dots\}$  for two choices of the sequence  $\{\xi_n, n = 1, 2, \dots\}$ .

The first case they considered was the case when

$$Pr[\xi_n = 0] = 1, \quad n = 1, 2, \dots$$

From a stochastic point of view this choice is somewhat artificial since in this case the sequence  $\{g^{(n)}(x), n = 1, 2, \dots\}$  is a deterministic sequence. Using results from the paper [1] by J.P. Keener, Lasota and Mackey concluded that when the parameters  $a$  and  $b$  in the expression (2) are chosen such that  $a = 1/2$  and  $b = 17/30$ , then the sequence  $\{g^{(n)}(x), n = 1, 2, \dots\}$  is a nonperiodic sequence for any initial value  $x$ . (For a more explicit proof of this fact see [4]; especially page 465.)

Lasota and Mackey then also considered the case when each of the stochastic variables  $\xi_n, n = 1, 2, \dots$  has a uniform distribution on the interval  $[0, 1/15]$ . Using computer simulations they observed that the distributions of the sequence  $g^{(n)}(\xi_0; \xi^{(n)})$ , where  $\xi_0$  has approximate uniform distribution on the interval  $[0, 1]$ , follow a 3-periodic pattern already for  $n \geq 10$ . (See [2], Figure 1 or [3], Figure 10.5.1.)

Thus, what Lasota and Mackey observed was that, although a function is such that it gives rise to a nonperiodic sequence of numbers when iterated, if - at each time epoch - the sequence of iterations is perturbed by a small stochastic number, then the distributions of the elements in the sequence may show a periodic pattern. They formulate this observation as follows:

"... However, the surprising content of Theorem 1 ( of [2]) is that even in a transformation  $S$  that has aperiodic limiting behaviour, the addition of noise will result in asymptotic periodicity.

This phenomenon is rather easy to illustrate numerically by considering...". (See [2], page 149.)

In the book [3] from 1994 by Lasota and Mackey, the authors also present the example described above. Part of the text in [3] concerning this example reads as follows:

"Thus, in this example (the example above) we have a noise induced period three asymptotic periodicity". (See [3], section 10.5, page 323.)

This observed transition from an aperiodic behaviour to a periodic behaviour - thanks to stochastic perturbations - is certainly an interesting observation. However this conclusion is not completely true in the sense that in the long run the 3-periodicity will slowly disappear. What holds is that for any initial value  $x$  the distributions of the process  $\{g^{(n)}(x, \xi^{(n)}), n = 1, 2, \dots\}$  will tend to a unique limit measure.

## 2 Motivation

Last year (2015), an interesting paper by F. Nakamura called *Periodicity of non-expanding piecewise linear maps and effects of random noises* was published (see [4]). Unfortunately though, in the last section of the paper, the author considers the stochastic process described above and makes the same claim as Lasota and Mackey. In fact, Nakamura even quotes the sentence from [3], that was mentioned above, verbatim.

It thus seems that still 29 years since the paper [2] was published and 22 years since the book [3] was published, the fact that the claim made by Lasota and Mackey concerning the limit behaviour of the distributions of the stochastic process described above is not correct, has not been pointed out in the literature. This is the motivation to write down a proof of the fact that the stochastic process considered by Lasota and Mackey in [2], section 5, and in [3] section 10.5, has a unique limit distribution.

The proof presented below is in principal quite straightforward and not difficult, but writing down all the details requires a few pages.

At this point it is worth mentioning that although the convergence rate to the unique limit measure is exponential - that is of order  $O(\rho^n)$  where  $\rho < 1$  - the parameter  $\rho$  is yet so close to unity that it is quite likely that it will not be possible to reach the limit distribution - nor even come close to the limit distribution - by computer simulations.

The observation made by Lasota and Mackey, that stochastic perturbation may induce a high degree of periodicity may certainly - under certain circumstances - be a useful and valuable observation.

## 3 Some simple formulas

For  $17/30 \leq b \leq 19/30$  define  $g_b : [0, 1) \rightarrow [0, 1)$

$$g_b(x) = x/2 + b \pmod{1}. \quad (3)$$

From (3) follows that

$$g_b(x) = x/2 + b, \text{ if } 0 \leq x < 2(1 - b)$$

$$g_b(x) = x/2 + b - 1 \text{ if } 2(1 - b) \leq x < 1.$$

Next define  $g_b^{(n)}$  recursively by  $g_b^{(1)} = g_b$ ,  $g_b^{(n+1)} = g_b \circ g_b^{(n)}$ . By simple calculations we find that  $g^{(2)}(x)$  satisfies

$$g_b^{(2)}(x) = x/4 + 3b/2, \text{ if } 0 \leq x < 4 - 6b,$$

$$g_b^{(2)}(x) = x/4 + 3b/2 - 1/2, \text{ if } 4 - 6b \leq x < 2(1 - b),$$

$$g^{(2)}(x) = x/4 + 3b/2 - 1, \text{ if } 2(1 - b) \leq x < 1,$$

and we find that  $g_b^{(3)}(x)$  satisfies

$$g_b^{(3)}(x) = x/8 + 7b/4, \text{ if } 0 \leq x \leq 8 - 14b \text{ and } 17/30 \leq b < 4/7,$$

$$g_b^{(3)}(x) = x/8 + 7b/4 - 1, \text{ if } \max\{0, 8 - 14b\} \leq x < 4 - 6b$$

$$g_b^{(3)}(x) = x/8 + 7b/4 - 1/2, \text{ if } 4 - 6b \leq x < 2(1 - b)$$

and

$$g_b^{(3)}(x) = x/8 + 7b/4 - 1/4, \text{ if } 2(1 - b) \leq x < 1.$$

Note that if  $b \geq 4/7$  then the set  $\{x : 0 \leq x < 8 - 14b\} = \emptyset$ .

Next set  $A = [17/30, 1)$  and let  $I_A : S \rightarrow \{0, 1\}$  denote the indicator function of  $A$ . The rotation number  $rot_{g_b}(x)$  of  $g_b$  can be defined by

$$rot_{g_b}(x) = \lim_{N \rightarrow \infty} N^{-1} \sum_{n=1}^N I_A(g_b^{(n)}(x))$$

(See [1], Definition 1.1, page 590.) Since  $g_b(0) > \lim_{x \rightarrow 1} g_b(x)$  it follows from Lemma 3.1 of [1] that  $rot_{g_b}(x)$  exists and is independent of  $x$ .

**Proposition 3.1** *If  $4/7 \leq b \leq 19/30$  then*

$$rot_{g_b}(x) = 1/3$$

*whereas if  $17/30 \leq b < 4/7$  then*

$$rot_{g_b}(x) < 1/3.$$

We shall not prove this proposition since it will not be used in our proof of Theorem 4.1 below. Let us just make a few observations.

- 1) If  $b = 4/7 = 120/210$  then  $g_b^{(3)}(0) = 0$ .
- 2) If  $b = 19/30$  and  $x_0 = 13/(7 \cdot 15)$  then  $0 < x_0 < 4 - 6(19/30)$  and

$$g_b^{(3)}(x_0) = x_0.$$

- 3) The ratio between the sets  $[17/30, 4/7] (= [119/210, 120/210])$  and  $[4/7, 19/30] (= [120/210, 133/210])$  is equal to  $1/13$ .

The first two observations indicate the truth of the proposition. The third observation, that the ratio between the sets  $[17/30, 4/7]$  and  $[4/7, 19/30]$  is equal to  $1/13$  and thus quite small, explains why computer simulations show a 3-periodic pattern. On the other hand, since the rotation number  $rot_{g_b}(x) < 1/3$  when  $17/30 < b < 4/7$  it is not surprising that in the long run the sequence  $\{g^{(n)}(x, \xi^{(n)}), n = 1, 2, \dots\}$  as defined in Section 1, has a unique limit measure independent of  $x$ , as we claimed above.

We shall end this section stating yet one more relation which gives some more information about the mapping  $g_b : [0, 1) \rightarrow [0, 1)$  when  $b = 17/30$ .

For, suppose that  $x = 26/30 - \epsilon$  where say for simplicity  $0 < \epsilon < 1/100$ . Then, by simple calculations, we find that

$$g_b^{(4)}(26/30 - \epsilon) = 26/30 - \epsilon/16,$$

if  $b = 17/30$  from which we see that  $g_b^{(4n)}(26/30 - \epsilon) \rightarrow 26/30$  as  $n \rightarrow \infty$  if  $b = 17/30$ , from which we can conclude that  $g_b$  is very close to a 4-periodic function if  $b = 17/30$ . That  $g_b$  is not a 4-periodic function when  $b = 17/30$  is easy to check by showing that the equation  $g_b^{(4)}(x) - x = 0$  has no solutions when  $b = 17/30$ .

## 4 A limit result

Let  $S = [0, 1)$ , let  $\delta : S \times S \rightarrow S$  be defined by

$$\delta(x, y) = |x - y|$$

and let  $\mathcal{B}$  be the Borel field on  $S$  determined by  $\delta$ . Further, as before let  $g : S \rightarrow S$  be defined by

$$g(x) = ax + b \pmod{1}.$$

where  $a = 1/2$  and  $b = 17/30$ .

Next let  $\Omega = [0, 2/30]$ , let  $\mathcal{A}$  be the Borel field on  $\Omega$ . Set  $\Omega^1 = \Omega$ ,  $\mathcal{A}^1 = \mathcal{A}$  and for  $n = 2, 3, \dots$  define  $\Omega^n$  and  $\mathcal{A}^n$  recursively by

$$\Omega^n = \Omega \times \Omega^{n-1}$$

$$\mathcal{A}^n = \mathcal{A}^{n-1} \otimes \mathcal{A}.$$

We denote a generic element in  $\Omega^n$  by  $\omega^n = (\omega_1, \omega_2, \dots, \omega_n)$ .

Next let  $\{f^{(n)} : S \times \Omega^n \rightarrow S, n = 1, 2, \dots\}$  be a sequence of functions defined recursively by

$$f^{(1)}(x, \omega) = g(x) + \omega \pmod{1}. \quad (4)$$

$$f^{(n+1)}(x, \omega^{n+1}) = f^{(1)}(f^{(n)}(x, \omega^n), \omega_{n+1}). \quad (5)$$

Let  $\{\xi_n, n = 1, 2, \dots\}$  be a sequence of independent, identically distributed, random variables having uniform distribution on the interval  $\Omega$  and set  $\xi^{(n)} = (\xi_1, \xi_2, \dots, \xi_n)$ . We denote the distribution of  $\xi_n$  by  $\lambda$  and the distribution of  $\xi^{(n)}$  by  $\lambda^n$ .

For  $n = 1, 2, \dots$  define  $K^n : S \times \mathcal{B} \rightarrow [0, 1]$  by

$$K^n(x, A) = Pr[f^{(n)}(x, \xi^{(n)}) \in A] = \int_A f^{(n)}(x, \omega^n) \lambda^n(d\omega^n). \quad (6)$$

**Theorem 4.1** *There exists a constant  $C > 0$ , a constant  $\rho < 1$  and a measure  $\mu$  such that for all  $x \in S$  and all  $A \in \mathcal{B}$*

$$|K^n(x, A) - \mu(A)| \leq C\rho^n. \quad (7)$$

The proof can be regarded as a "routine matter". Our proof is based on a simple *coupling device*.

## 5 An auxiliary limit theorem for Markov chains

Let  $(S, \mathcal{F}, \delta)$  be a compact metric space where  $\mathcal{F}$  is the Borel field induced by the metric  $\delta$ . Let  $P : S \times \mathcal{F} \rightarrow [0, 1]$  be a transition probability function (tr.pr.f). Let  $P^n : S \times \mathcal{F} \rightarrow [0, 1]$  denote the  $n$ -step tr.pr.f induced by  $P : S \times \mathcal{F} \rightarrow [0, 1]$ . Let  $\mathcal{P}(S, \mathcal{F})$  denote the set of probability measures on  $(S, \mathcal{F})$ . If  $\mu, \nu \in \mathcal{P}(S, \mathcal{F})$  we let  $\|\mu - \nu\|$  denote the total variance distance between  $\mu$  and  $\nu$  defined as usual by

$$\|\mu - \nu\| = \sup\{\mu(F) - \nu(F) : F \in \mathcal{F}\} + \sup\{\nu(F) - \mu(F) : F \in \mathcal{F}\}$$

and we let  $\tilde{\mathcal{P}}(S^2, \mathcal{F}^2, \mu, \nu)$ , denote the set of all **couplings** of  $\mu$  and  $\nu$ ; that is the set of all probability measures  $\tilde{\mu}$  on  $(S \times S, \mathcal{F} \otimes \mathcal{F})$  such that

$$\tilde{\mu}(F \times S) = \mu(F), \forall F \in \mathcal{F}$$

and

$$\tilde{\mu}(S \times F) = \nu(F), \forall F \in \mathcal{F}.$$

We say that a tr.pr.f  $\tilde{P} : S^2 \times \mathcal{F}^2 \rightarrow [0, 1]$  is a **Markovian coupling** of  $P : S \times \mathcal{F} \rightarrow [0, 1]$  if for each  $x, y \in S$ ,  $\tilde{P}(x, y, \cdot)$  is a coupling of  $P(x, \cdot)$  and  $P(y, \cdot)$ .

**Definition 5.1** *We say that  $P : S \times \mathcal{F} \rightarrow [0, 1]$  has the **overlapping property** if there exists a set  $S_0 \in \mathcal{F}$  such that*

*1) there exist an integer  $N$  and a number  $\alpha_1 > 0$  such that*

$$\inf_{x \in S} P^N(x, S_0) \geq \alpha_1$$

*2) there exist a number  $\alpha_2 > 0$  and a Markovian coupling  $\tilde{P}_0 : S^2 \times \mathcal{F}^2 \rightarrow [0, 1]$  of  $P$  such that if  $D = \{(x, y) \in S \times S : x = y\}$  then*

$$\inf\{\tilde{P}_0((x, y), D) : x, y \in S_0\} \geq \alpha_2$$

*If we want to emphasize the parameters involved in the definition of the overlapping property, we say that  $P : S \times \mathcal{F} \rightarrow [0, 1]$  has the overlapping property with basic set  $S_0$ , basic integer  $N_0$ , basic coupling  $\tilde{P}_0 : S^2 \times \mathcal{F}^2 \rightarrow [0, 1]$  and basic lower bounds  $\alpha_1$  and  $\alpha_2$ .*

The following limit result holds.

**Theorem 5.1** *Let  $(S, \mathcal{F}, \delta)$  be a compact metric space. Suppose  $P : S \times \mathcal{F} \rightarrow [0, 1]$  has the overlapping property. Then there exists a constant  $C > 0$ , a constant  $0 < \rho < 1$  and a probability measure  $\mu \in \mathcal{P}(S, \mathcal{F})$ , such that*

$$\sup\{||P^n(x, \cdot) - P^n(y, \cdot)|| : x, y \in S\} \leq C\rho^n, n = 1, 2, \dots \quad (8)$$

and

$$\sup\{||P^n(x, \cdot) - \mu|| : x \in S\} \leq C\rho^n, n = 1, 2, \dots \quad (9)$$

This theorem is not difficult to prove but for sake of completeness we give a proof in the appendix.

**Corollary 5.1** *In order to prove Theorem 4.1 it suffices to prove that the tr.pr.f  $K : [0, 1) \times \mathcal{B} \rightarrow [0, 1]$  defined by*

$$K(x, F) = Pr[f^{(1)}(x, \xi) \in F], \quad (10)$$

*where  $f^{(1)}$  is defined by (4) and  $\xi$  is uniformly distributed on  $[0, 1/15]$ , has the overlapping property.*

**Proof.** In order to be able to use Theorem 4.1 we need to verify that  $K^n : S \times \mathcal{B} \rightarrow [0, 1]$ , as defined by (6), is in fact the  $n$ -step tr.pr.f induced by the tr.pr.f  $K : S \times \mathcal{B} \rightarrow [0, 1]$  defined by (10). But this follows easily from the definition of  $\{f^{(n)} : S \times \Omega^n \rightarrow S, n = 1, 2, \dots\}$ . (See (4) and (5).)  $\square$

## 6 Determining a basic set.

In order to prove that the tr.pr.f  $K : [0, 1) \times \mathcal{B} \rightarrow [0, 1]$  defined by (10) has the overlapping property we shall first prove the following proposition.

**Proposition 6.1** *Let*

$$S_0 = [0, 3/30]$$

*and*

$$D = \{(x, y) \in S \times S : x = y\}.$$

*Let  $K : [0, 1) \times \mathcal{B} \rightarrow [0, 1]$  be defined as in Corollary 5.1. Then we can find a Markovian coupling  $\tilde{K} : S^2 \times \mathcal{F}^2 \rightarrow [0, 1]$  such that*

$$\inf\{\tilde{K}((x, y), D) : x, y \in S_0\} \geq 1/4$$

**Proof.** We divide  $S \times S$  into four disjoint sets as follows.

$$S_1 = \{(x, y) \in S \times S : 0 \leq (y - x)/2 \leq 2/30\},$$

$$S_2 = \{(x, y) \in S \times S : 2/30 < (y - x)/2\},$$

$$S_3 = \{(x, y) \in S \times S : 0 < (x - y)/2 \leq 2/30\}$$

*and*

$$S_4 = \{(x, y) \in S \times S : 2/30 < x - y)/2\}.$$

As before let  $g(x) = x/2 + 17/30 \pmod{1}$ .

Next we define  $h_1 : S \times S \times \Omega \rightarrow S$  by

a)

$$h_1(x, y, \omega) = g(x) + \omega + (y - x)/2 \pmod{1}$$

if

$$(x, y) \in S_1 \text{ and } \omega + (y - x)/2 \leq 2/30,$$

b)

$$h_1(x, y, \omega) = g(x) + \omega + (y - x)/2 - 2/30 \pmod{1}$$

if

$$(x, y) \in S_1 \text{ and } \omega + (y - x)/2 > 2/30,$$

and c)

$$h_1(x, y, \omega) = g(x) + \omega \pmod{1}$$

if

$$(x, y) \in S_2 \cup S_3 \cup S_4,$$

and we define  $h_2 : S \times S \times \Omega \rightarrow S$  by

a)

$$h_2(x, y, \omega) = g(y) + \omega + (x - y)/2 \pmod{1}$$

if

$$(x, y) \in S_3 \text{ and } \omega + (x - y)/2 \leq 2/30,$$

b)

$$h_2(x, y, \omega) = g(y) + \omega + (x - y)/2 - 2/30 \pmod{1}$$

if

$$(x, y) \in S_3 \text{ and } \omega + (x - y)/2 > 2/30,$$

and finally c)

$$h_2(x, y, \omega) = g(y) + \omega \pmod{1}$$

if

$$(x, y) \in S_1 \cup S_2 \cup S_4.$$

We also define  $\tilde{h} = (\tilde{h}_1, \tilde{h}_2) : S \times S \times \Omega \rightarrow S \times S$  by

$$\tilde{h}_1(x, y, \omega) = h_1(x, y, \omega)$$

and

$$\tilde{h}_2(x, y, \omega) = h_2(x, y, \omega),$$

and we define  $\tilde{K} : S \times S \times \mathcal{B} \otimes \mathcal{B} \rightarrow [0, 1]$  by

$$\tilde{K}(x, y, F) = \lambda\{\omega : \tilde{h}(x, y, \omega) \in F\} \quad (11)$$

**Lemma 6.1** *The function  $\tilde{K} : S \times S \times \mathcal{B} \otimes \mathcal{B} \rightarrow [0, 1]$  defined above has the following properties.*

- a)  $\tilde{K} : S \times S \times \mathcal{B} \otimes \mathcal{B} \rightarrow [0, 1]$  is a tr.pr.f,
- b)  $\tilde{K} : S \times S \times \mathcal{B} \otimes \mathcal{B} \rightarrow [0, 1]$  is a Markovian coupling of the tr.pr.f  $K : S \times \mathcal{B} \rightarrow [0, 1]$  defined by (6),
- c) if  $x, y \in S_0$  and  $D = \{(x, y) \in S \times S : x = y\}$  then

$$\tilde{K}(x, y, D) \geq 1/4. \quad (12)$$

**Proof.** That  $\tilde{K}(x, y, \cdot) \rightarrow [0, 1]$  is a probability measure for every  $(x, y) \in S \times S$  follows easily from the definition of  $\tilde{K} : S \times S \times \mathcal{B} \otimes \mathcal{B} \rightarrow [0, 1]$ . (See (11).) That also  $\tilde{K}(\cdot, \cdot, F) \rightarrow [0, 1]$  is  $\mathcal{B} \otimes \mathcal{B}$ -measurable if  $F = A \times B$ , where  $A$  and  $B$  are intervals, follows easily from the definitions of  $h_1 : S \times S \times \Omega \rightarrow S$  and  $h_2 : S \times S \times \Omega \rightarrow S$ , and since the set of all rectangular sets  $A \times B$  is a base for  $\mathcal{B} \otimes \mathcal{B}$ , it follows that  $\tilde{K}(\cdot, \cdot, F) \rightarrow [0, 1]$  is  $\mathcal{B} \otimes \mathcal{B}$ -measurable for every  $F \in \mathcal{B} \otimes \mathcal{B}$ . Thus  $\tilde{K} : S \times S \times \mathcal{B} \otimes \mathcal{B} \rightarrow [0, 1]$  is a tr.pr.f which proves part a) of the lemma.

Next let us consider  $\tilde{K}(x, y, A \times S)$  for  $A \in \mathcal{B}$ . From the definition of  $\tilde{K} : S \times S \times \mathcal{F} \otimes \mathcal{F} \rightarrow [0, 1]$  (see (11)) it follows that

$$K(x, y, A \times S) = \lambda\{\omega : h_1(x, y, \omega) \in A\}.$$

If  $(x, y) \in S_2 \cup S_3 \cup S_4$ , then  $h_1(x, y, \omega) = g(x) + \omega \pmod{1}$  from which immediately follows that in this case  $K(x, y, A \times S) = K(x, A)$ .

We also have to consider the case when  $(x, y) \in S_1$ . In this case

$$h_1(x, y, \omega) = g(x) + (y - x)/2 + \omega \pmod{1}$$

if  $(y - x)/2 + \omega \leq 2/30$  and

$$h_1(x, y, \omega) = g(x) + (y - x)/2 + \omega - 2/30 \pmod{1}$$

if  $(y - x)/2 + \omega > 2/30$ . Now, if  $A \in \mathcal{B}$ , and for each  $z \in [0, 2/30]$  we define  $A_z = \{\omega \in \Omega : g(x) + z + \omega \in A \text{ and } z + \omega < 2/30\} \cup \{\omega \in \Omega : g(x) + z + \omega - 2/30 \in A \text{ and } z + \omega - 2/30 \geq 0\}$  it follows easily that  $\lambda(A) = \lambda(A_z)$  from which follows that  $\tilde{K}(x, y, A \times S) = K(x, A)$  also in this case.



That  $\tilde{K}(x, y, S \times A) = K(y, A)$ ,  $\forall A \in \mathcal{B}$  can be proved in a similar way. Thereby part b) of the lemma is proved.

It remains to prove part c). But, if  $x, y \in S_0$  then

$$|y - x|/2 \leq 1/20.$$

Suppose first that  $x \leq y$ . We then find that

$$\tilde{h}_2(x, y, \omega) = g(y) + \omega = y/2 + 17/30 + \omega.$$

We also find that if also  $0 \leq (y - x)/2 + \omega \leq 2/30$  then

$$\tilde{h}_1(x, y, \omega) = g(x) + (y - x)/2 + \omega = x/2 + 17/30 + (y - x)/2 + \omega = 17/30 + y/2 + \omega =$$

$$g(y) + \omega = \tilde{h}_2(x, y, \omega).$$

Hence if  $x, y \in S_0$ ,  $x \leq y$  and  $(y - x)/2 + \omega \leq 2/30$  then

$$\tilde{h}_1(x, y, \omega) = \tilde{h}_2(x, y, \omega).$$

But clearly, since  $0 \leq (y - x)/2 \leq 1/20 < 2/30$

$$\lambda\{\omega : (y - x)/2 + \omega \leq 2/30\} = 15(2/30 - (y - x)/2) \geq 15(2/30 - 1/20) =$$

$$15(4 - 3)/60 = 1/4,$$

from which follows that (12) holds if  $x, y \in S_0$  and  $0 \leq x \leq y$ . That (12) holds also if  $x, y \in S_0$  and  $0 \leq y < x$  can be proved similarly. Thereby also part c) of Lemma 6.1 is proved and from Lemma 6.1 follows Proposition 6.1.  $\square$ .

## 7 Finding return times for elements in the basic set

In the previous section we verified one of the two hypotheses that the tr.pr.f  $K : S \times \mathcal{B} \rightarrow [0, 1]$  has to fulfill in order to have the overlapping property. (See Definition 5.1.) It thus remains to verify that we can find an integer  $N$  and a number  $\alpha$  such that

$$\inf_{x \in S} K^N(x, S_0) \geq \alpha,$$

where thus  $S_0 = [0, 3/30]$ .

As a first step we shall in this section prove the following proposition.

**Proposition 7.1** *As above, for  $n = 1, 2, \dots$ , let  $K^n : S \times \mathcal{B} \rightarrow [0, 1]$  be defined by (6) and let  $S_0 = [0, 3/30]$ . There exist constants  $\alpha_0 > 0$  and  $\beta_0 > 0$  such that*

$$\inf_{x \in S_0} K^3(x, S_0) \geq \alpha_0 \tag{13}$$

and

$$\inf_{x \in S_0} K^7(x, S_0) \geq \beta_0. \tag{14}$$

**Proof.** Let

$$T_0 = \{x : 0 \leq x \leq 1/45\}$$

and

$$\Omega_0^3 = \{\omega^3 = (\omega_1, \omega_2, \omega_3) \in \Omega^3 : \omega_1/4 + \omega_2/2 + \omega_3 < 1/180\}.$$

As before, for  $n = 1, 2, \dots$ , let  $f^{(n)} : S \times \Omega^n \rightarrow S$  be defined by (4) and (5). Then, by simple calculations, we find that

$$119/120 < f^{(3)}(x, \omega^3) = x/8 + 119/120 + \omega_1/4 + \omega_2/2 + \omega_3 < 1$$

if  $x \in T_0$  and  $\omega^3 \in \Omega_0^3$ . Hence, if we define  $T_1 = \{x : 119/120 \leq x < 1\}$  we find that if  $x \in T_0$  then

$$Pr[f^{(3)}(x, \xi^{(3)}) \in T_1] \geq Pr[\xi^{(3)} \in \Omega_0^3]$$

and since

$$Pr[\xi^{(3)} \in \Omega_0^3] = (4/3) \cdot 15^3 \cdot (1/180)^3 = (4/3) \cdot (1/12)^3 = (1/6)^4 = 1/1296,$$

we find that

$$Pr[f^{(3)}(x, \xi^{(3)}) \in T_1] \geq 1/1296$$

if  $x \in T_0$ . Furthermore, since

$$f(x, \omega) \in S_0$$

if  $x \in T_1$  and  $\omega \leq 1/30$ , we find that

$$Pr[f^{(1)}(x, \xi_1) \in S_0] \geq 1/2$$

if  $x \in T_1$  and hence

$$Pr[f^{(4)}(x, \xi^{(4)}) \in S_0] \geq (1/1296) \cdot (1/2) = 1/2592 \quad (15)$$

if  $x \in T_0$ .

Next, let  $x \in S_0$  and define

$$\Omega_x^3 = \{\omega^3 = (\omega_1, \omega_2, \omega_3) \in \Omega^3 : x/8 + 119/120 + \omega_1/4 + \omega_2/2 + \omega_3 \geq 1\}$$

and define

$$W_x^3 = \{\omega^3 = (\omega_1, \omega_2, \omega_3) \in \Omega_x^3 : x/8 - 1/120 + \omega_1/4 + \omega_2/2 + \omega_3 < 1/45\}.$$

Then, since

1)

$$f^{(3)}(x, \omega^3) = x/8 + 119/120 + \omega_1/4 + \omega_2/2 + \omega_3 - 1$$

if  $x \in S_0$  and  $\omega^3 \in \Omega_x^3 \cap W_x^3$ ,

2)

$$-1/120 \leq x/8 - 1/120 \leq 1/240 \text{ if } x \in S_0$$

and 3)

$$1/45 - 1/240 = 13/720 > 12/720 = 1/60,$$

it is not difficult to convince oneself that

$$Pr[f^{(3)}(x, \xi^{(3)}) \in T_0] \geq (4/3) \cdot 15^3 \cdot (1/60)^3 = 1/48 \quad (16)$$

if thus  $x \in S_0$ .

Furthermore, since by monotonicity,

$$\sup_{x \in S_0} Pr[f^{(3)}(x, \xi^{(3)}) \in T_1] = \sup_{x \in T_0} Pr[f^{(3)}(x, \xi^{(3)}) \in T_1] =$$

$$Pr[f^{(3)}(0, \xi^{(3)}) \in T_1] = (4/3)15^3(1/120)^3 = 1/384$$

it follows that we must have

$$Pr[f^{(3)}(x, \xi^{(3)}) \in S_0] \geq 383/384$$

if  $x \in S_0$ .

By combining (16) and (15) and using the fact that

$$f^{(7)}(x, \xi^{(7)}) = f^{(3)}(f^{(4)}(x, \xi^{(4)}), (\xi_5, \xi_6, \xi_7))$$

it follows from the Markov property that

$$Pr[f^{(7)}(x, \xi^{(7)}) \in S_0] \geq (1/48)(1/2592) = 1/(3^5 \cdot 2^9) > 1/125000 = 8 \cdot 10^{-6}$$

if  $x \in S_0$ . Hence, setting

$$\alpha_0 = 383/384 \quad (17)$$

and

$$\beta_0 = 1/(3^5 \cdot 2^9) \quad (18)$$

we find that (13) and (14) hold and thereby Proposition 7.1 is proved.  $\square$

**Corollary 7.1** *Let  $\alpha_0$  and  $\beta_0$  be defined (17) and (18) respectively, let  $K : S \times \mathcal{B} \rightarrow [0, 1]$  be defined as in Corollary 5.1. Then, if  $x \in S_0$*

$$K^{12}(x, S_0) \geq \alpha_0^4$$

$$K^{13}(x, S_0) \geq \alpha_0^2 \beta_0$$

$$K^{14}(x, S_0) \geq \beta_0^2$$

**Proof.** Follows from (13), (14) and the Markov property.  $\square$ .

**Corollary 7.2** *Let  $\alpha_0$  and  $\beta_0$  be defined (17) and (18) respectively, let  $K : S \times \mathcal{B} \rightarrow [0, 1]$  be defined as in Corollary 5.1. Then for every  $n \geq 12$  there exists a number  $\gamma_n > 0$  such that if  $x \in S_0$*

$$K^n(x, S_0) \geq \gamma_n.$$

**Proof.** Follows from Corollary 7.1, (13) and the Markov property.  $\square$

## 8 First entrance time to the basic set

In the previous section we showed that

$$Pr[f^{(n)}(x, \xi^{(n)}) \in S_0] > 0$$

for all  $n \geq 12$  if  $x \in S_0$ . In this section we shall investigate  $K^n(x, S_0)$  when  $x \notin S_0$ .

We have already proved that

$$K(x, S_0) \geq 1/2 \text{ if } x \in T_1 \quad (19)$$

where thus  $T_1 = [119/120, 1)$ .

Next set  $T_2 = [24/30, 119/120]$ . Since  $f(24/30, \omega) = 12/30 + 17/30 + \omega - 1$  if  $\omega \geq 1/30$  and  $f(24/30, 2/30) = 1/30 < 3/30$  we find that  $f(24/30, \omega) \in S_0$  if  $\omega > 1/30$  and since  $f(119/120, \omega) \in S_0$  if  $0 \leq \omega < 1/30 + 1/240$ , we can conclude easily that

$$K(x, S_0) \geq 1/2 \text{ if } x \in T_2. \quad (20)$$

Next set  $T_3 = [12/30, 24/30]$ . It is easily seen that in this case

$$K(x, T_2 \cup T_1) \geq 1/2.$$

Hence

$$K^2(x, S_0) \geq 1/4, \text{ if } x \in T_3. \quad (21)$$

It remains to consider the interval  $T_4 = [3/30, 12/30]$ . This time it is easily seen that

$$K(x, T_3) \geq 1/2,$$

and consequently

$$K^3(x, S_0) \geq 1/8, \text{ if } x \in T_4 \quad (22)$$

Combining (19), (20), (21) and (22) with Corollary 7.1, we can conclude that

$$\inf_{x \in S} K^{15}(x, S_0) \geq (1/2)\beta_0^2 \approx 3 \cdot 10^{-11},$$

where thus  $\beta_0 = 3^{-5} \cdot 2^{-9} \approx 8 \cdot 10^{-6}$ . Thereby we have verified that  $K : S \times \mathcal{B} \rightarrow [0, 1]$  has the overlapping property and hence Theorem 4.1 follows from Theorem 5.1.  $\square$

## 9 Appendix 1. Proof of Theorem 5.1

The purpose of this appendix is to prove Theorem 5.1. For sake of convenience we repeat the formulation.

**Theorem 5.1.** *Let  $(S, \mathcal{F}, \delta)$  be a compact metric space. Suppose  $P : S \times \mathcal{F} \rightarrow [0, 1]$  has the overlapping property. Then there exists a constant  $C > 0$ , a constant  $0 < \rho < 1$  and a probability measure  $\mu$  such that*

$$\sup\{\|P^n(x, \cdot) - P^n(y, \cdot)\| : x, y \in S\} \leq C\rho^n, \quad n = 1, 2, \dots,$$

and

$$\sup\{\|P^n(x, \cdot) - \mu\| : x \in S\} \leq C\rho^n, \quad n = 1, 2, \dots$$

**Proof.** Let  $B[S, \mathcal{F}]$  denote the bounded, real, Borel-measurable functions on  $(S, \mathcal{F})$ . For  $u \in B[S, \mathcal{F}]$  define

$$\|u\| = \sup\{|u(x)| : x \in S\}$$

and

$$\text{osc}(u) = \sup\{u(x) - u(y) : x, y \in S\}.$$

For  $u \in B[S, \mathcal{F}]$  and  $\mu \in \mathcal{P}(S, \mathcal{F})$  we write

$$\int_S u(x) \mu(dx) = \langle u, \mu \rangle.$$

Next, let  $\mu, \nu \in \mathcal{P}(S, \mathcal{F})$ . It is well-known that

$$\|\mu - \nu\| = \sup\{\langle u, \mu \rangle - \langle u, \nu \rangle : u \in B[S, \mathcal{F}], \|u\| \leq 1\}. \quad (23)$$

Thus, what we need to prove is that there exists a constant  $C$  and a number  $0 < \rho < 1$ , such that for  $x, y \in S$

$$\sup\{\langle u, P^n(x, \cdot) \rangle - \langle u, P^n(y, \cdot) \rangle : u \in B[S, \mathcal{F}], \|u\| \leq 1\} < C\rho^n, n = 1, 2, \dots$$

We start our proof with the following lemma.

**Lemma 9.1** *Let  $\mu, \nu \in \mathcal{P}(S, \mathcal{F})$  and suppose that there exists a coupling  $\tilde{\mu}$  of  $\mu$  and  $\nu$  such that*

$$\tilde{\mu}(D) = \alpha > 0$$

*where as above  $D = \{(x, y) \in S \times S : x = y\}$ . Let  $u \in B[S, \mathcal{F}]$ . Then*

$$|\langle u, \mu \rangle - \langle u, \nu \rangle| \leq (1 - \alpha)\text{osc}(u).$$

**Proof.** Let us first point out that the diagonal set  $D$  belongs to the  $\sigma$ -field  $\mathcal{F} \otimes \mathcal{F}$  since  $(S, \mathcal{F}, \delta)$  is a compact metric space. Next let  $u \in B[S, \mathcal{F}]$ . Then

$$\begin{aligned} \left| \int_S u(x) \mu(dx) - \int_S u(x) \nu(dx) \right| &= \left| \int_{S \times S} (u(x) - u(y)) \mu(dx) \nu(dy) \right| = \\ &= \left| \int_{S \times S} (u(x) - u(y)) \tilde{\mu}(dx, dy) \right| \leq \\ &= \left| \int_{(S \times S) \setminus D} (u(x) - u(y)) \tilde{\mu}(dx, dy) \right| + \left| \int_D (u(x) - u(y)) \tilde{\mu}(dx, dy) \right| \leq (1 - \alpha)\text{osc}(u). \end{aligned}$$

**Corollary 9.1** *Let  $P : S \times \mathcal{F} \rightarrow [0, 1]$  be the tr.pr.f of Theorem 5.1. Since  $P$  has the overlapping property there exist a basic set  $S_0$ , a basic Markovian coupling  $\tilde{P}_0$  and a constant  $\alpha_2 > 0$  such that*

$$\inf\{\tilde{P}_0(x, y, D) : x, y \in S_0\} \geq \alpha_2$$

*Let  $x, y \in S_0$ . Then*

$$\|P(x, \cdot) - P(y, \cdot)\| \leq 1 - \alpha_2.$$

**Proof.** Follows from Lemma 9.1 and (23).  $\square$

**Corollary 9.2** Let  $P, S_0, \tilde{P}_0$  and  $\alpha_2 > 0$  be as in Corollary 9.1, and let  $\mu, \nu \in \mathcal{P}(S, \mathcal{F})$  be such that

$$\mu(S_0) \geq \alpha$$

and

$$\nu(S_0) \geq \alpha.$$

Define  $\mu_1 \in \mathcal{P}(S, \mathcal{F})$  by

$$\mu_1(F) = \int_S P(x, F) \mu(dx), \quad F \in \mathcal{F}$$

and  $\nu_1 \in \mathcal{P}(S, \mathcal{F})$  by

$$\nu_1(F) = \int_S P(x, F) \nu(dx), \quad F \in \mathcal{F}.$$

Then

$$\|\mu_1 - \nu_1\| \leq 1 - \alpha_2 \cdot \alpha^2.$$

**Proof.** Define  $\tilde{\mu}_1 \in \mathcal{P}(S^2, \mathcal{F}^2)$  by

$$\tilde{\mu}_1(A) = \int_{S \times S} \tilde{P}_0(x, y, A) \mu(dx) \nu(dy).$$

It is easily checked that  $\tilde{\mu}_1$  is a coupling of  $\mu_1$  and  $\nu_1$ . Furthermore we find that

$$\begin{aligned} \tilde{\mu}_1(D) &= \int_{S \times S} \tilde{P}_0(x, y, D) \mu(dx) \nu(dy) = \int_{S_0 \times S_0} \tilde{P}_0(x, y, D) \mu(dx) \nu(dy) + \\ &\quad \int_{(S \times S) \setminus (S_0 \times S_0)} \tilde{P}_0(x, y, D) \mu(dx) \nu(dy) \geq \alpha^2 \alpha_2 + 0. \end{aligned}$$

From Lemma 9.1 now follows that

$$|\langle u, \mu_1 \rangle - \langle u, \nu_1 \rangle| \leq (1 - \alpha^2 \alpha_2) \text{osc}(u)$$

if  $u \in B[S, \mathcal{F}]$ , which implies that

$$\|\mu_1 - \nu_1\| \leq (1 - \alpha_2 \cdot \alpha^2). \quad \square$$

**Corollary 9.3** Let  $P : S \times \mathcal{F} \rightarrow [0, 1]$  have the overlapping property with basic set  $S_0$ , basic integer  $N_0$ , basic coupling  $\tilde{P}_0 : S^2 \times \mathcal{F}^2 \rightarrow [0, 1]$  and basic lower bounds  $\alpha_1$  and  $\alpha_2$ . Then

$$\|P^{N_0+1}(x, \cdot) - P^{N_0+1}(y, \cdot)\| \leq (1 - \alpha_1^2 \alpha_2), \quad \forall x, y \in S.$$

**Proof.** Let  $x, y \in S$ . Since

$$P^{N_0}(z, S_0) \geq \alpha_2, \quad \forall z \in S$$

it is clear that

$$P^{N_0}(x, S_0) \geq \alpha_2,$$

and

$$P^{N_0}(y, S_0) \geq \alpha_2.$$

Since  $P^{N_0+1}(x, \cdot) \in \mathcal{P}(S, \mathcal{F})$  is defined by

$$P^{N_0+1}(x, F) = \int_S P(z, F) P^{N_0}(x, dz)$$

and similarly  $P^{N_0+1}(y, \cdot) \in \mathcal{P}(S, \mathcal{F})$  is defined by

$$P^{N_0+1}(\cdot, F) = \int_S P(z, F) P^{N_0}(y, dz)$$

we see that the hypotheses of Corollary 9.2 are satisfied. The conclusion of Corollary 9.3 now follows from Corollary 9.2.  $\square$

Next, let  $T : B[S, \mathcal{F}] \rightarrow B[S, \mathcal{F}]$  be defined by

$$Tu(x) = \int_S u(y) P(x, dy)$$

where thus  $P$  has the properties of the theorem under consideration. If  $u \in B[S, \mathcal{F}]$  we may write

$$T^m u = u_m$$

if convenient.

Next set  $N_1 = N_0 + 1$  and  $\rho_1 = 1 - \alpha_1^2 \alpha_2$ . From Corollary 9.3 it follows that

$$\sup\{T^{N_1}u(x) - T^{N_1}u(y) : x, y \in S\} \leq \rho_1 \text{osc}(u)$$

for all  $u \in B[S, \mathcal{F}]$ . Hence, for  $m = 1, 2, \dots$

$$\text{osc}(T^{N_1+m}) \leq \rho_1 \text{osc}(u_m).$$

By induction it follows that

$$\text{osc}(T^{kN_1}) \leq \text{osc}(u) \rho_1^k, \quad k = 1, 2, \dots$$

Since also  $\text{osc}(T^n u) \leq \text{osc}(u)$ ,  $\forall u \in B[S, \mathcal{F}]$  we conclude that

$$\text{osc}(T^n u) \leq C \rho^n \text{osc}(u), \tag{24}$$

for all  $u \in B[S, \mathcal{F}]$  if  $\rho$  and  $C$  are defined by

$$\rho = (\rho_1)^{1/N_1}$$

$$C = 1/\rho,$$

and since (24) holds for all  $u \in B[S, \mathcal{F}]$ , the estimate (8) also holds and thereby the first conclusion of Theorem 5.1 is proved. (See (8).)

That also the second inequality of Theorem 5.1 holds, follows easily from the first as follows. First, since  $\text{osc}(T^n(u)) \rightarrow 0$  and  $(S, \mathcal{F}, \delta)$  is supposed to be a compact metric space, it follows that there exists a unique, invariant measure  $\mu$ , such that

$$\lim_{n \rightarrow \infty} \int_S u(y) P^n(x, dy) - \langle u, \mu \rangle = 0, \quad \forall x \in S.$$

Furthermore, if we define  $Q : \mathcal{P}(S, \mathcal{F}) \rightarrow \mathcal{P}(S, \mathcal{F})$  by

$$Q\nu(A) = \int_S P(x, A)\nu(dx), \quad \forall A \in \mathcal{F}$$

and use the fact that if  $u \in B[K, \mathcal{F}]$  and  $\nu \in \mathcal{P}(K, \mathcal{F})$  then

$$\langle Tu, \nu \rangle = \langle u, Q\nu \rangle$$

and the fact that  $\mu = Q\mu$  since  $\mu$  is invariant, we find that if  $u \in B[S, \mathcal{F}]$  then for  $n = 1, 2, \dots$  we have

$$\begin{aligned} & \left| \int_S u(y)P^n(x, dy) - \langle u, \mu \rangle \right| = \\ & \left| \int_S (T^n u(x) - T^n u(y))\mu(dy) \right| \end{aligned}$$

which together with (24) implies that for all  $x \in S$

$$\left| \int_S u(y)P^n(x, dy) - \langle u, \mu \rangle \right| \leq \text{osc}(u)C\rho^n$$

which implies that

$$\sup\{\|P^n(x, \cdot) - \mu\| : x \in S\} \leq C\rho^n, \quad n = 1, 2, \dots$$

and thereby also the second conclusion of Theorem 5.1 is proved. (See (9).)  $\square$

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